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A GEOMETRICAL APPROACH TO THE CONSUBSTANTIALITY OF THE GRAVITATIONAL AND ELECTROMAGNETIC INTERACTION

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ABSTRACT
In this article, we establish that the gravitational and electromagnetic fields have a consubstantial nature, and obtain expressions for the description electromagnetic-gravitational field in Y^4 space. The applied approach can be described as follows. First, we developed Eddington's ideas to obtain an analog of Maxwell's theory in Y^4 interpreted the symmetric terms as metric the antisymmetric part as an electromagnetic field. Then, the antisymmetric part is studied as an electromagnetic field, finally, the Weyl theory is introduced to determine the geometrical structure of the world by applying the Weyl assumption about partitionbetween geometry and electricity.
Keywords: Electromagnetism, Gravity, Geometry, Eddington's theory, Unified Field Theory,torsion.
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INTRODUCTION
After the theory of general relativity was developed between 1907 and 1920 arose a problem of unification of the gravitational and electromagnetic fields. There are several different approaches to the creation of unified field theory, the most recent is a string theory that considers the high-dimension spaces with following its reduction (convolution) to the observable phenomenal space; the initial approach was made by founding fathers of relativity theory A. Einstein [10-14], A. Eddington [9],and H. Weyl [36-38], its main idea is to construct four-dimensional continuum so that the mathematical description of all fundamental forces can be coherently obtained from the variational principle, this theory must contain gravitational and Maxwell's field equations in a natural way [5-6, 43].

In [41-47] were studied the geometrical structure of the Y^n space and some of its applications to the physical problems. In the present article, we consider the amalgamations ofA. Einstein, A. Eddington, and H. Weyl ideas in one comprehensive theory based on the geometrical structure of space with torsion, where the connection is defined by

$$\Gamma^p_{kl} = \frac{1}{2} g^{pi}\left(g_{ik,l} + g_{li,k} - g_{kl,i} + g_{km}S^m_{li} + g_{lm}S^m_{ki}\right) + \frac{1}{2}S^p_{kl}.$$

Let us recall that the classical Maxwell equations in four-dimensional representations describe the electromagnetic phenomena of the physical world. The Maxwell electromagnetic field can be described in terms of a four-vector potential $A_i = \left(\phi, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3\right)$ in the form

$$F_{ik} = A_{i;k} - A_{k;i} = A_{i,k} - A_{k,i} + S^p_{ki}A_p.$$

Maxwell's equations are

$$F_{ik,l} + F_{li,k} + F_{kl,i} = 0$$

and

$$F^{ik}{}_{;k} = \frac{4\pi}{c} J^i,$$

here $J^i = \left(\rho, J^1, J^2, J^3\right)$ is a vector of current. The electromagnetic field in a vacuum can be express as

$$F^{ik}{}_{;k} = \mathbf{0}.$$

The main idea of this article is to combine different relativistic methods into one coherent consistent model of the electromagnet gravitational theory in the Riemannian space with torsion, which provides a most simple possible description of gravitational and electromagnetic fields and reveals their consubstantiality.

THE FIELD EQUATIONS IN Y^n SPACE

Let us consider a four-dimensional continuum with the structure of Y^n the manifold.
In this space, the infinitesimal parallel transportation of vector A^i defines according to formula

$$A^i{}_{;k} = A^i{}_{,k} + \Gamma^i{}_{jk} A^j, \qquad (1)$$

where we denoted $\frac{\partial A^i}{\partial x^k} = A^i{}_{,k}$ and can be shown that connection can be calculated by using metric and torsion tensors [33] as

$$\Gamma^p_{kl} = \frac{1}{2} g^{pi}\left(g_{ik,l} + g_{li,k} - g_{kl,i} + g_{km}S^m_{li} + g_{lm}S^m_{ki}\right) + \frac{1}{2}S^p_{kl}. \quad (2)$$

The tensor R^p_{ikl} is curvature tensor of Y^n and can be written as [33]

$$R_{ik} = R^p_{pik} = \Gamma^p_{ip,k} - \Gamma^p_{ik,p} + \Gamma^p_{qk}\Gamma^q_{ip} - \Gamma^p_{qp}\Gamma^q_{ik}$$

and

$$R_{ik} = R^p_{pik} = \Gamma^p_{ip,k} - \Gamma^p_{ik,p} + \Gamma^p_{qk}\Gamma^q_{ip} - \Gamma^p_{qp}\Gamma^q_{ik}, \qquad (3)$$

Where the connection can be written in the form $\Gamma^p_{kl} = \mathrm{P}^p_{kl} + L^p_{kl}$ or $\Gamma^p_{kl} - \mathrm{P}^p_{kl} = L^p_{kl}$.
From the definitions of covariant derivative and curvature tensor, we have an equation

$$S^i_{jk;p;q} - S^i_{jk;q;p} = R^t_{qpj}S^i_{tk} + R^t_{qpk}S^i_{jt} - R^i_{qpt}S^t_{jk} + S^t_{qp}S^i_{jk;t}$$

or

$$S^i_{jk;p;q} - S^i_{jk;q;p} - S^t_{qp}S^i_{jk;t} = R^t_{qpj}S^i_{tk} + R^t_{qpk}S^i_{jt} - R^i_{qpt}S^t_{jk}.$$

EDDINGTON'S APPROACH

we assume that the connection of the space does not depend on the space metric.We compose the scalar density as

$$W_g = \sqrt{-g}\, g^{ik} R_{ik}$$

and are applying the variation principle of the least action postulate that all variation of the functional

$$\int g^{ik} R_{ik} \sqrt{-g}\, dV$$

with respect to the connection equals zero, the integral not to be varied at the boundaries.

By standard calculations, we have

$$\delta \int g^{ik} R_{ik}\sqrt{-g}\, dV =$$

$$= \int g^{ik}\delta\left(\Gamma^p_{ip,k} - \Gamma^p_{ik,p} + \Gamma^p_{qk}\Gamma^q_{ip} - \Gamma^p_{qp}\Gamma^q_{ik}\right)\sqrt{-g}\, dV,$$

the variation with respect to the connection Γ^m_{nt} gives

$$\int\left(g^{nt}{}_{,m} - g^{nk}{}_{,k}\delta^t_m + g^{it}\Gamma^n_{im} + g^{nk}\Gamma^t_{mk} - g^{nt}\Gamma^p_{mp} - g^{ik}\Gamma^n_{ik}\delta^t_m\right)\sqrt{-g}\,\delta\left(\Gamma^m_{nt}\right) dV = \mathbf{0}$$

and

$$g^{nt}{}_{,m} - g^{nk}{}_{,k}\delta^t_m + g^{it}\Gamma^n_{im} + g^{nk}\Gamma^t_{mk} - g^{nt}\Gamma^p_{mp} - g^{ik}\Gamma^n_{ik}\delta^t_m = \mathbf{0}.$$

By contracting this equation by indices t and m, we have obtained

$$g^{nm}{}_{,m} - \mathbf{4}g^{nk}{}_{,k} + g^{im}\Gamma^n_{im} + g^{nk}\Gamma^m_{mk} - g^{nm}\Gamma^p_{mp} - \mathbf{4}g^{ik}\Gamma^n_{ik} = \mathbf{0}$$

and

$$\mathbf{3}\left(g^{nm}{}_{,m} + g^{im}\Gamma^n_{im}\right) + g^{nm}S^p_{mp} = \mathbf{0}. \qquad (4)$$

We contract (5) with tensor g_{nt} and obtain

$$g_{nt}g^{nt}{}_{,m} - g_{nm}g^{nk}{}_{,k} + \Gamma^n_{nm} - 3\Gamma^n_{mn} - g_{nm}g^{ik}\Gamma^n_{ik} = 0 \qquad (5)$$

or

$$2g^{pm}\left(\left(\ln\left(\sqrt{g}\right)\right)_{,m} + \Gamma^n_{mn}\right) + g^{pm}{}_{,m} + g^{pm}S^n_{mn} + g^{im}\Gamma^p_{im} = 0.$$

We introduce the notations and obtain

$$\gamma^n = \frac{1}{3}g^{nm}S^p_{mp} = -\left(g^{nm}{}_{,m} + g^{im}\Gamma^n_{im}\right) =$$
$$= 2g^{nm}\left(\left(\ln\left(\sqrt{g}\right)\right)_{,m} + \Gamma^p_{mp}\right) + g^{nm}S^p_{mp} = -g^{nm}\left(\left(\ln\left(\sqrt{g}\right)\right)_{,m} + \Gamma^p_{mp}\right). \qquad (6)$$

The field equations [44]

Let us contract the equation

$$S^i_{jk;p;q} - S^i_{jk;q;p} = R^t_{qpj}S^i_{tk} + R^t_{qpk}S^i_{jt} - R^i_{qpt}S^t_{jk} + S^t_{qp}S^i_{jk;t}, \qquad (7)$$

by the indices k, p and multiply by g^{js}, we have

$$\left(g^{kp}g^{js}S^i_{sk;p} - g^{kp}g^{js}S^i_{qp}S^q_{sk}\right)_{;i} - g^{kp}g^{js}S^i_{sk;i;p} - g^{kp}g^{js}S^i_{pq;i}S^q_{sk} =$$
$$= g^{kp}g^{js}R^t_{ips}S^i_{tk} + g^{kp}g^{js}R^t_{ipk}S^i_{st} - g^{kp}g^{js}R^i_{ipt}S^t_{sk}. \qquad (8)$$

Let us denote

$$H^{ji} = g^{kp}g^{js}S^i_{sk;p} - g^{kp}g^{js}S^i_{qp}S^q_{sk}, \qquad (9)$$

and

$$F^{jp} = g^{kp}g^{js}S^i_{sk;i}. \qquad (10)$$

We introduce an asymmetric in any pair of indices tensor

$$C^{ijk} = g^{pj}g^{qk}S^i_{pq} + g^{pk}g^{qi}S^j_{pq} + g^{pi}g^{qj}S^k_{pq} \qquad (11)$$

(12)

We have

$$H^{ji}_{;i} - F^{ji}_{;i} - g^{kp}g^{js}S^q_{sk}F_{pq} = g^{kp}g^{js}R^t_{ips}S^i_{tk} + g^{kp}g^{js}R^t_{ipk}S^i_{st} - g^{kp}g^{js}R^i_{ipt}S^t_{sk}, \qquad (13)$$

Where $F_{pq} = S^i_{pq;i}$.

We are obtaining

$$H^{jk} - H^{kj} = C^{ikj}_{;i} + F^{jk} + g^{kp}g^{qs}S^t_{pq}S^j_{ts} - g^{jp}g^{qs}S^t_{pq}S^k_{ts}, \qquad (14)$$

we have that

$$g^{kp}g^{qs}S^t_{pq}S^j_{ts} - g^{jp}g^{qs}S^t_{pq}S^k_{ts} = \frac{1}{2}\left(C^{jpq}S^k_{pq} - C^{kpq}S^j_{pq}\right), \qquad (15)$$

Hence

$$H^{jk} - H^{kj} = C^{ikj}_{;i} + F^{jk} + \frac{1}{2}\left(C^{jpq}S^k_{pq} - C^{kpq}S^j_{pq}\right). \qquad (16)$$

Easy to show that

$$C^{ikj}_{;i} = -C^{ijk}_{;i} = -\left(C^{ijk}_{,i} + \Gamma^j_{pi}C^{ipk} + \Gamma^k_{pi}C^{ijp} + \Gamma^i_{pi}C^{pkj}\right). \qquad (17)$$

We conclude

$$\Gamma^j_{pi}C^{ipk} = \Gamma^j_{ip}C^{ipk} = \frac{1}{2}\left(\Gamma^j_{ip}C^{ipk} + \Gamma^j_{pi}C^{pik}\right) = \frac{1}{2}S^j_{ip}C^{pki} = -\frac{1}{2}S^j_{pi}C^{kpi}$$

and

$$\Gamma^k_{pi}C^{ijp} = \Gamma^k_{ip}C^{pji} = \frac{1}{2}C^{jpi}\left(\Gamma^k_{ip} - \Gamma^k_{pi}\right) = \frac{1}{2}S^k_{ip}C^{jpi} = \frac{1}{2}S^k_{pq}C^{jqp}.$$

Then, we have

$$C^{ikj}_{;i} = -C^{ijk}_{;i} = -C^{ijk}_{,i} - \frac{1}{2} S^{j}_{pq} C^{kpq} + \frac{1}{2} S^{k}_{pq} C^{jpq} - \Gamma^{q}_{pq} C^{pkj}, \quad (18)$$

and we conclude

$$H^{jk} - H^{kj} = -C^{ijk}_{,i} - \Gamma^{q}_{pq} C^{pkj} + F^{jk}. \quad (19)$$

Since $\Gamma^{p}_{pl} = \frac{1}{2} g_{ip,l} g^{ip} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^{l}}, \Gamma^{p}_{lp} = \Gamma^{p}_{pl} + S^{p}_{lp}$

and $\Gamma^{p}_{pl} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^{l}} + (\ln \psi)_{,l} = \left(\ln\left(\psi \sqrt{-g}\right)\right)_{,l}$,

we obtain

$$H^{jk} - H^{kj} - F^{jk} = -C^{ijk}_{,i} - \left(\ln\left(\psi \sqrt{-g}\right)\right)_{,i} C^{ikj}. \quad (20)$$

Finally, we have obtained equality

$$\left(\psi \sqrt{-g} \left(H^{jk} - H^{kj} - F^{jk}\right)\right)_{,k} = 0, \quad (21)$$

where $S^{p}_{ip} = \varphi_{i} = 3 g_{in} \gamma^{n} = (\ln \psi)_{,i}$.

Since

$$\gamma^{n} = \frac{1}{3} g^{nm} S^{p}_{mp} = -g^{nm} \left(\left(\ln\left(\sqrt{g}\right)\right)_{,m} + \Gamma^{p}_{mp} \right) \quad (22)$$

and recalling that $\Gamma^{p}_{pl} = \frac{1}{2} g_{ip,l} g^{ip} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^{l}}$, $\Gamma^{p}_{lp} = \Gamma^{p}_{pl} + S^{p}_{lp}$,

and $\Gamma^{p}_{lp} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^{l}} + (\ln \psi)_{,l} = \left(\ln\left(\psi \sqrt{-g}\right)\right)_{,l}$, we obtain

$$\gamma^{n} = \frac{1}{3} g^{nm} S^{p}_{mp} = -g^{nm} \left(\left(\ln\left(\sqrt{g}\right)\right)_{,m} + \left(\ln\left(\psi \sqrt{-g}\right)\right)_{,m} \right). \quad (23)$$

The difference $L^{p}_{kl} - \frac{1}{2} S^{p}_{kl}$ is a symmetrical tensor and following Eddington's idea, we can obtain

$$L^{p}_{kl} - \frac{1}{2} S^{p}_{kl} = \frac{1}{6} \delta^{p}_{k} J_{l} + \frac{1}{6} \delta^{p}_{l} J_{k} - \frac{1}{2} g_{kl} g^{pq} J_{q}, \quad (24)$$

where we hypothesize $\gamma^{n} = J^{n}$.

THE WEYL THEORY

Contracted Riemanean tensor can be represented in the form of the sum of two tensors symmetric and antisymmetric

$$R_{ik} = s_{ik} + a_{ik}, \quad (25)$$

Where tensors s_{ik} *and* a_{ik} are functions of the connection

Symmetric tensor s_{ik} corresponds to the gravitational field (can be understood as a metric tensor) and an antisymmetric tensor a_{ik} corresponds to the electromagnetic field.

From these tensors and densities, we could compose the scalar density $\wp$ and applying the variational principle to the Hamiltonian integral and taking into considerations that the variations of the field to vanish on the boundary of the variational domain, we could obtain

$$\int \left(\frac{\partial \wp}{\partial s_{ik}} \delta s_{ik} + \frac{\partial \wp}{\partial a_{ik}} \delta a_{ik} \right) dV = 0. \quad (26)$$

Let $\mathbf{f}^{ik}$ be the density of the electromagnetic field and let us denote

$$\frac{\partial \wp}{\partial s_{ik}} = \mathbf{g}^{ik}$$
$$\frac{\partial \wp}{\partial a_{ik}} = \mathbf{f}^{ik}, \qquad (27)$$

The tensor densities $\mathbf{g}^{ik}, \mathbf{f}^{ik}$ are the densities of gravitational and electromagnetic fields.

By variational principle, we are obtaining

$$2\mathbf{g}^{nt}{}_{;m} - \mathbf{g}^{nk}{}_{;k}\delta^t_m - \mathbf{g}^{tk}{}_{,k}\delta^n_m - \frac{\partial \mathbf{f}^{nk}}{\partial x_k}\delta^t_m - \frac{\partial \mathbf{f}^{tk}}{\partial x_k}\delta^n_m = \mathbf{0}. \qquad (28)$$

Next, we are using duality argument and obtain

$$\int \left(\frac{\partial \wp^*}{\partial s_{ik}} \delta \mathbf{g}^{ik} + \frac{\partial \wp^*}{\partial a_{ik}} \delta \mathbf{f}^{ik} \right) dV = \mathbf{0} \qquad (29)$$

and

$$\frac{\partial \wp^*}{\partial \mathbf{g}^{ik}} = s_{ik}$$
$$\frac{\partial \wp^*}{\partial \mathbf{f}^{ik}} = a_{ik}. \qquad (30)$$

The function-vector $\frac{\partial \mathbf{f}^{nk}}{\partial x_k}$ is a density of electric current and $g_{ik} = s_{ik}$.

The conditions (27) means that the expression in the brackets under the integral (30) is an exact differential of s_{ik} *and* a_{ik} as variables, similarly (30) states that the expression under the integral symbol (29) is an exact differential of $\mathbf{g}^{ik}$ *and* $\mathbf{f}^{ik}$ as arguments.

CONCLUSIONS

An amalgamation of these three aspects of the description of the natural phenomena gives us a possibility to obtain a comprehensive relativistic theory, which provides mathematical apparatuses for the representation of physical understanding of physical phenomena of the electromagnetic and gravitational fields from a single viewpoint and gives a possibility to deduct electromagnetic and gravitational field equations as its special cases.

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